Grade 7/8 Math Circles
February 6/7/8/9, 2023
Circle Geometry

The History of Circles

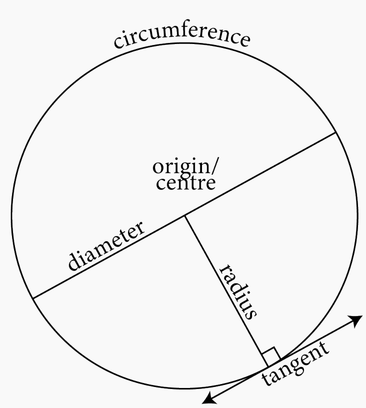
Circles have been used for millennia: as religious symbols, as wheels on carts, and certainly in math. We don't have a specific date for when circles were discovered, because they existed even before recorded history. The earliest mathematical theorems about circles are attributed to Thales from around 650 BCE, while other prominent Greek mathematicians who have worked on circles include Euclid and Archimedes, who was the first one to figure out an approximation for π as $\pi = \frac{22}{7} \approx 3.142857$.

What is π?

Before answering this question, let us first define some basic circle terminology.

Definitions

Origin/Centre: The middle of a circle.
Circumference: The perimeter of a circle.
Radius: The distance from the origin to the edge of a circle.
Diameter: The distance from one edge of a circle to another through the origin. It's also twice the length of the radius.
Tangent: A line that touches the circumference of the circle at only one point.



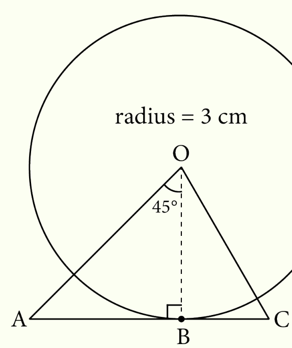
Important Fact: The tangent is perpendicular to the radius passing through the same point.

You might know that the formula for the circumference is $C = 2\pi r$ (where C represents the circumference and r represents the radius), but have you ever thought about what it means? When we look at an equation like $a = 2b$, we know that a is twice the value of b. We call 2 the ratio between a and b. So in the case of $C = 2\pi r$, we know that the circumference is 2π times the size of r or that half of the circumference is π times the size of r. Why is this important? This means that π is the ratio between the circumference and 2 times the radius (or between the circumference and diameter). What an interesting connection!

Check out this interactive tool to explore this idea more.

Exercise 1

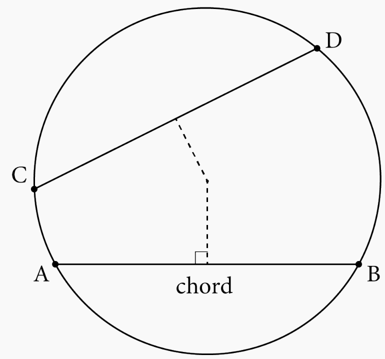
Given that $\overline{AC} = 5$ cm, and the radius of the circle is 3 cm, find the value of $\overline{BC}$ using your knowledge of circles and triangles. (Hint: You might want to use the fact that the sum of the interior angles of a triangle is 180°.)



Chords

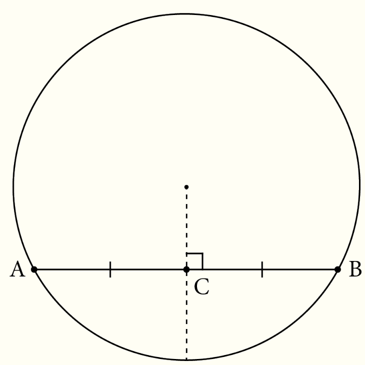
Definition

Chord: A line from one edge of a circle to another (not necessarily through the origin).



A key property of a chord is that the radius bisects it (divides it evenly into two). This gives us our first theorem.

Theorem 1 (Radius Bisects Chord)**.** *The radius, when perpendicular to the chord, will bisect it.*



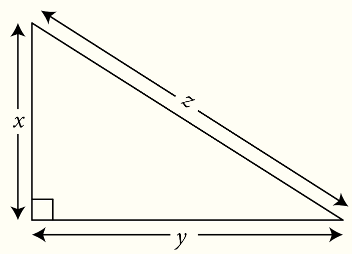
Explore

Stop and Think

Suppose we wanted to find the length of a chord, given the radius and perpendicular distance between the chord and the origin. Use the diagram given with the definition of a chord to reason about what the formula for the length of a chord might be. (Hint: You'll need to use the Pythagorean Theorem.)

Pythagorean Theorem

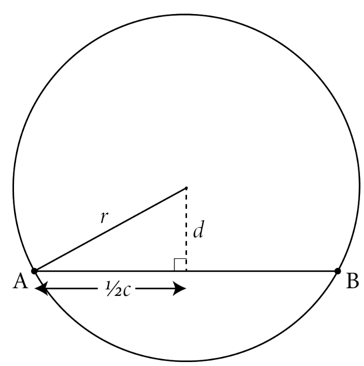
For any right angled triangle, $x^2 + y^2 = z^2$.



The formula for the length of a chord is actually:

$$\text{chord length} = 2\sqrt{\text{radius}^2 - (\text{distance between chord and origin})^2}.$$

This makes sense because we can form a right angled triangle with half of the chord ($\frac{1}{2}c$) by **Theorem 1**, the perpendicular distance (d) between the chord and origin, and the distance between the origin and the end point of the chord (r).

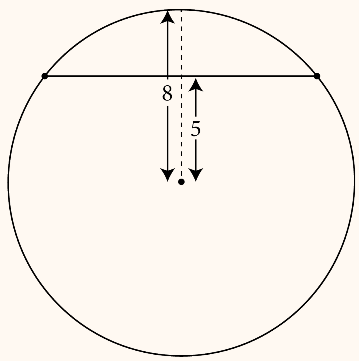


This allows us to apply the Pythagorean Theorem to get the following result:

$$\begin{aligned} r^2 &= d^2 + (\frac{1}{2}c)^2 \\ r^2 - d^2 &= (\frac{1}{2}c)^2 \\ \frac{1}{2}c &= \sqrt{r^2 - d^2} \\ c &= 2\sqrt{r^2 - d^2} \end{aligned}$$

Example A

Calculate the chord length given that $d = 5$ and $r = 8$.



Solution

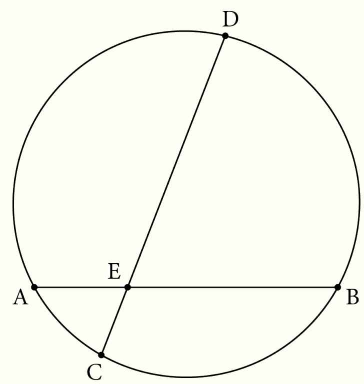
Since we are given $d = 5$ and $r = 8$, we only need to substitute these values into the formula for the chord length.

$$\begin{aligned} c &= 2\sqrt{r^2 - d^2} \\ &= 2\sqrt{8^2 - 5^2} \\ &= 2\sqrt{64 - 25} \\ &= 2\sqrt{39} \\ &\approx 12.49 \end{aligned}$$

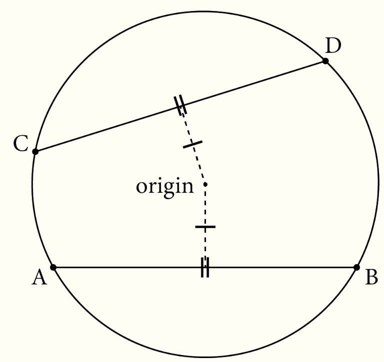
Chord Theorems

Now let's explore a couple of theorems that are useful to help us solve problems that use chords.

Theorem 2 (Intersecting Chords Theorem)**.** *Suppose $\overline{AB}$ and $\overline{CD}$ are intersecting chords as shown in the diagram below. Then $\overline{AE} \times \overline{BE} = \overline{CE} \times \overline{DE}$.*



Theorem 3 (Equal Chords and Equidistant from Center Theorem)**.** *Suppose $\overline{AB} = \overline{CD}$. Then the chords are equidistant (same distance) from the origin.*



Stop and Think

Is the opposite the same for **Theorem 3** (ie. If the chords are equidistant from the origin, then the chords are the same length)?

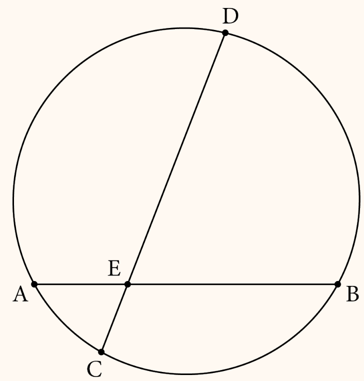
Yes! If the chords are equidistant from the origin, then they must also be equal in length. The equation for the length of a chord, depends only on the radius and the distance from the origin. Since both the radius and the distance from the origin are the same for any two chords that are equidistant from the origin, this must mean that the chord lengths must also be equal.

Exercise 2

1. Given two chords of equal length, which theorem might you apply?
   (a) Theorem 1 (Radius Bisects Chord)
   (b) Theorem 2 (Intersecting Chords Theorem)
   (c) Theorem 3 (Equal Chords and Equidistant from Center Theorem)
   (d) None of the above
2. Given that a chord bisects another, which theorem might you apply?
   (a) Theorem 1 (Radius Bisects Chord)
   (b) Theorem 2 (Intersecting Chords Theorem)
   (c) Theorem 3 (Equal Chords and Equidistant from Center Theorem)
   (d) None of the above
3. Given two chords that intersect at their end point, which theorem might you apply?
   (a) Theorem 1 (Radius Bisects Chord)
   (b) Theorem 2 (Intersecting Chords Theorem)
   (c) Theorem 3 (Equal Chords and Equidistant from Center Theorem)
   (d) None of the above

Example B

Given that $\overline{AB} = 6$ cm, $\overline{AE} = 2$ cm, and $\overline{CE} = 1.6$ cm, what is the length of $\overline{CD}$?



Solution

Since the two chords intersect at point E, we can use **Theorem 1** (Intersecting Chords Theorem). The theorem states that $\overline{AE} \times \overline{BE} = \overline{CE} \times \overline{DE}$. Rearranging, we get:

$$\overline{AE} \times \overline{BE} = \overline{CE} \times \overline{DE}$$
$$\overline{DE} = \frac{\overline{AE} \times \overline{BE}}{\overline{CE}}$$

Since $\overline{AB} = 6$ cm and $\overline{AE} = 2$ cm, we know that $\overline{BE} = 6 \text{ cm} - 2 \text{ cm} = 4$ cm. Substituting in the known values gives:

$$\overline{DE} = \frac{2 \times 4}{1.6} \text{ cm}$$
$$\overline{DE} = 5 \text{ cm}$$

Finally, since $\overline{CD} = \overline{CE} + \overline{DE}$, we get that $\overline{CD} = 1.6 + 5 = 6.6$ cm.

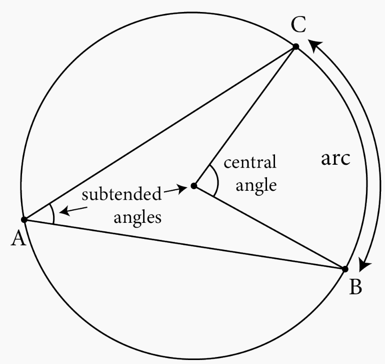
Arcs and Angles

Definitions

Arc length: A portion of the perimeter of a circle.

Subtended angle: The angle created by two intersecting lines.

Central angle: The subtended angle at the origin.



In this diagram, we say that $\angle BAC$ is subtended by $\overline{AC}$ and $\overline{AB}$ at point A.

Explore

What does an arc remind you of? If you were thinking the crust of something (possibly pie or pizza), then I would agree with you. The crust of a pie or pizza represents the edge of a slice, and a slice is essentially a fraction of the total. When you were learning fractions, you probably explored the idea by drawing slices of a whole. We know that the formula for the circumference of a circle is $C = 2\pi r$.

Stop and Think

Keeping in mind that an arc length is a proportion of the total circumference, what might the formula for an arc length be? (Hint: Think about the central angle of a full circle versus the central angle of an arc length.)

If we think of the arc length as a proportion of the circumference, we want to relate the proportion of the circumference to another measurement that we can take the proportion of. Angles make it very easy for us, since the angle of a full circle does not depend on anything; it will always be 360°. So the proportion for the central angle of the arc to the central angle of a full circle would be $\frac{\text{arc's central angle}}{360^\circ}$.

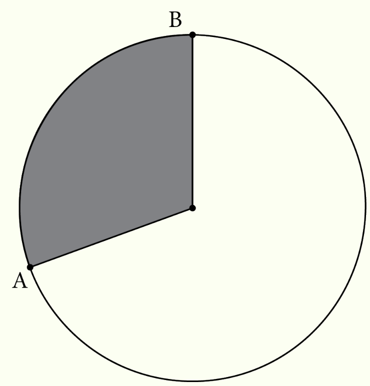
So taking this proportion, we can multiply it by the circumference to get:

$$L = \frac{\theta}{360^\circ} \times 2\pi r$$

where L denotes the arc length and θ denotes the arc angle.

Exercise 3

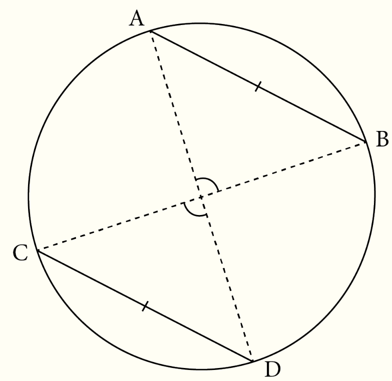
Given that the area of the shaded region is a third of the total area, what would the central angle of $\widehat{AB}$ (the arc between A and B) be? If the radius of the circle is 9 m, what would the length of $\widehat{AB}$ be?



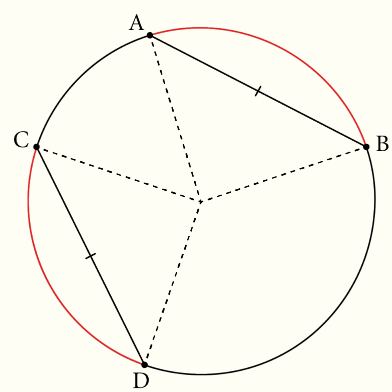
Note that we will use $\widehat{AB}$ to denote the length of the arc between A and B going forward.

Arc Theorems

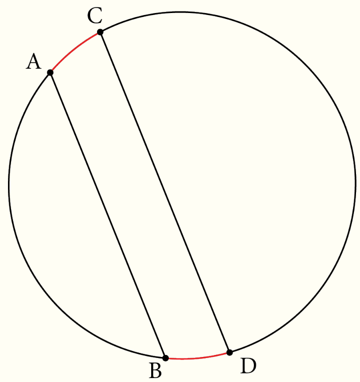
Theorem 4 (Chord Central Angle Theorem). *If $\overline{AB} = \overline{CD}$, then the two central angles subtended from the two chords are also congruent.*



Theorem 5 (Chord Arcs Theorem). *If $\overline{AB} = \overline{CD}$, then $\overset{\frown}{AB} = \overset{\frown}{CD}$.*



Theorem 6 (Parallel Chords Intercepted Arcs Theorem). *If $\overline{AB} \parallel \overline{CD}$ ($\overline{AB}$ and $\overline{CD}$ are parallel), then $\widehat{AC} = \widehat{BD}$.*

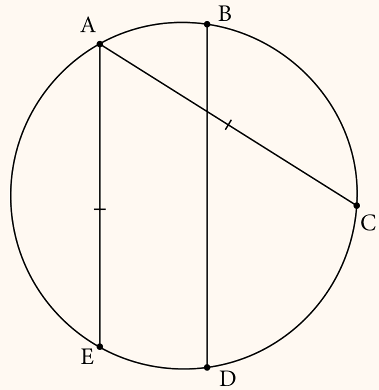


Exercise 4

1. Given an angle and a radius, which theorem might you apply?
   (a) Theorem 4 (Chord Central Angle Theorem)
   (b) Theorem 5 (Chord Arcs Theorem)
   (c) Theorem 6 (Parallel Chords Intercepted Arcs Theorem)
   (d) None of the above
2. Given two parallel chords, which theorem might you apply?
   (a) Theorem 4 (Chord Central Angle Theorem)
   (b) Theorem 5 (Chord Arcs Theorem)
   (c) Theorem 6 (Parallel Chords Intercepted Arcs Theorem)
   (d) None of the above
3. Given two chords of the same length, which theorem might you apply?
   (a) Theorem 4 (Chord Central Angle Theorem)
   (b) Theorem 5 (Chord Arcs Theorem)
   (c) Theorem 6 (Parallel Chords Intercepted Arcs Theorem)
   (d) None of the above

Example C

In the following diagram, $\overline{AE}$ and $\overline{BD}$ are parallel and $\overline{AE} = \overline{AC}$. Also, given that $\overset{\frown}{AE} = 5$ m and $\overset{\frown}{DE} = 2$ m, find $\overset{\frown}{BC}$ (the arc length between B and C).



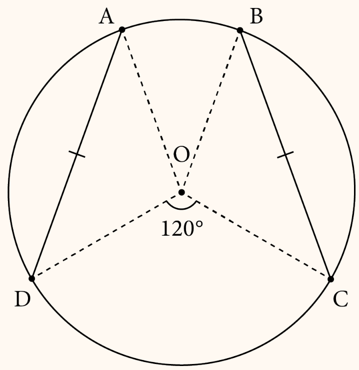
Solution

Since $\overline{AE} = \overline{AC}$, we can apply **Theorem 5** (Chord Arcs Theorem) to conclude that $\overset{\frown}{AE} = \overset{\frown}{AC}$. Also, since $\overline{AE} \parallel \overline{BD}$, we can apply **Theorem 6** (Parallel Chords Intercepted Arcs Theorem) to conclude that $\overset{\frown}{AB} = \overset{\frown}{DE}$. Then, notice that the sum of arcs $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ give us the arc $\overset{\frown}{AC}$. So we have the following:

$$\overset{\frown}{AC} = \overset{\frown}{AB} + \overset{\frown}{BC}$$
$$\overset{\frown}{BC} = \overset{\frown}{AC} - \overset{\frown}{AB}$$
$$\overset{\frown}{BC} = \overset{\frown}{AE} - \overset{\frown}{AB} \qquad \text{Applying } \textbf{Theorem 5}$$
$$\overset{\frown}{BC} = \overset{\frown}{AE} - \overset{\frown}{DE} \qquad \text{Applying } \textbf{Theorem 6}$$
$$\overset{\frown}{BC} = 5 \text{ m} - 2 \text{ m}$$
$$\overset{\frown}{BC} = 3 \text{ m}$$

Example D

In the following diagram, $\overline{AD} = \overline{BC}$. Given that $\angle COD = 120°$ and $\angle BOC = 100°$, find $\angle AOB$.



Solution

Since $\overline{AD} = \overline{BC}$, we can apply Theorem 4 (Chord Central Angle Theorem) to conclude that $\angle AOD = \angle BOC$. Notice that the full circle is cut into four angles: $\angle AOB$, $\angle BOC$, $\angle COD$, and $\angle AOD$. Since we know that the angle of a full circle is 360°, we can find $\angle AOB$ by subtracting the other three known angles. This gives:

$$360^\circ = \angle AOB + \angle BOC + \angle COD + \angle AOD$$
$$\angle AOB = 360^\circ - \angle BOC - \angle COD - \angle AOD$$
$$\angle AOB = 360^\circ - 2\angle BOC - \angle COD \qquad \text{Applying } \textbf{Theorem 4}$$
$$\angle AOB = 360^\circ - 2(100^\circ) - 120^\circ$$
$$\angle AOB = 40^\circ$$

Combining Arcs, Angles, and Chords

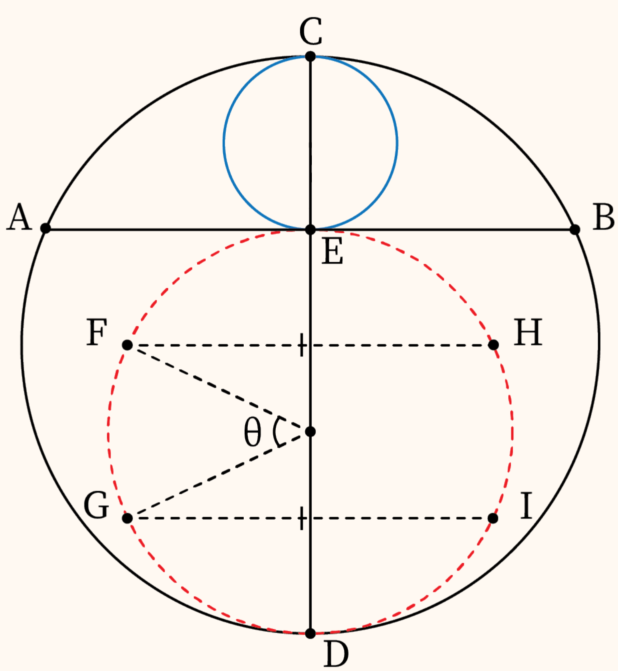
Now that we've talked about arcs, angles, chords, and a few theorems, let's try some more difficult examples where we will mix and and match these ideas to solve different geometric problems.

Example E

Suppose we call the black outer circle Circle 1, the dotted red circle Circle 2, and the small blue circle Circle 3. Also, suppose the following are true:

- the radius of circle 2 is 7.5 m
- $\theta = 50°$
- $\overline{CD}$ is a daimeter of Circle 1
- $\overline{AE} = 10$ m
- $\overline{FH} = \overline{GI}$
- $\widehat{GI} = \frac{65\pi}{12}$ m

How much larger is Circle 1's circumference compared to $\widehat{EG}$?



Solution

To solve this problem, we must first notice a few things. First, notice that since $\overline{CD}$ bisects Circle 1, we know that $\overline{AE} = \overline{BE}$ by **Theorem 1**. This is because $\overline{CD}$ is the diameter of Circle 1, and therefore the line segment from the centre of Circle 1 to C is the radius. Second, notice that since $\overline{FH} = \overline{GI}$, we know that $\widehat{FH} = \widehat{GI}$ by **Theorem 5**. Third, notice that $\overline{AB}$ intersects $\overline{CD}$ at point E. Both are chords, so we can apply **Theorem 2**. Lastly, notice that the diameter of Circle 1 is the sum of the diameters of the two smaller circles.

So, we break our solution into 5 steps:

1. Calculate the arc length between F and G
2. Find the radius of Circle 1
3. Calculate the circumference of Circle 1
4. Find the arc length between E and G
5. Compare the arc length between E and G and the circumference of Circle 1

Step 1: Calculate the arc length between F and G

Apply the arc length formula with the known values $\theta = 50°$ and radius of Circle 2 = 7.5 m.

$$\begin{aligned}\widehat{FG} &= \frac{\theta}{360°} \times 2\pi r \\ &= \frac{50°}{360°} \times 2\pi(7.5) \text{ m} \\ &= \frac{5}{36} \times 2\pi(7.5) \text{ m} \\ &= \frac{25\pi}{12} \text{ m} \\ &\approx 6.54 \text{ m}\end{aligned}$$

Step 2: Find the radius of Circle 1

To find the radius of Circle 1, note that it can be found by summing the radii (plural of radius) of Circle 2 and Circle 3, since the diameter of Circle 1 is the sum of the diameters of Circles 2 and 3 and the diameter of a circle is twice the radius. The value of the radius of Circle 2 is already given, so we only have to find the radius of Circle 3, which we can do by applying

Theorem 1 and finding $\overline{CE}$.

$$\begin{aligned}
\overline{AE} \times \overline{BE} &= \overline{CE} \times \overline{DE} \\
\overline{AE} \times \overline{AE} &= \overline{CE} \cdot 2(\text{Circle 2 radius}) \quad \text{Applying } \textbf{Theorem 3} \\
(10)(10) &= \overline{CE} \cdot 2(7.5) \text{ m} \\
\overline{CE} &= \frac{100}{15} \text{ m} \\
\overline{CE} &= \frac{20}{3} \text{ m} \\
\overline{CE} &\approx 6.66 \text{ m}
\end{aligned}$$

Since $\overline{CE}$ is the diameter of Circle 3, we know that $\overline{CE} = 2(\text{Circle 3 radius})$, which implies that $(\text{Circle 3 radius}) = \frac{10}{3}$ m. Now that we have the radii of Circle 2 and Circle 3, we sum them to get the radius of Circle 1.

$$\begin{aligned}
(\text{Circle 1 radius}) &= (\text{Circle 2 radius}) + (\text{Circle 3 radius}) \\
&= 7.5 + \frac{10}{3} \text{ m} \\
&= \frac{45}{6} + \frac{20}{6} \text{ m} \\
&= \frac{65}{6} \text{ m} \\
&\approx 10.83 \text{ m}
\end{aligned}$$

Step 3: Calculate the circumference of Circle 1

Apply the circumference formula with the known value $(\text{Circle 1 radius}) = \frac{65}{6}$ m

$$\begin{aligned}
(\text{Circle 1 circumference}) &= 2\pi(\text{Circle 1 radius}) \\
&= 2\pi\left(\frac{65}{6} \text{ m}\right) \\
&= \frac{65\pi}{3} \text{ m} \\
&\approx 68.07 \text{ m}
\end{aligned}$$

Step 4: Find the arc length between E and G

Notice that $\overset{\frown}{EG} = \overset{\frown}{EF} + \overset{\frown}{FG}$. We already know $\overset{\frown}{FG}$ from step 1, so we only need to find $\overset{\frown}{EF}$. We noted above that $\overset{\frown}{FH} = \overset{\frown}{GI}$ by Theorem 5, and $\overset{\frown}{EF}$ is half of $\overset{\frown}{FH}$. So we have the following:

$$\begin{aligned}\overset{\frown}{EG} &= \overset{\frown}{EF} + \overset{\frown}{FG} \\ &= \frac{1}{2}\overset{\frown}{FH} + \overset{\frown}{FG} \\ &= \frac{1}{2}\overset{\frown}{GI} + \overset{\frown}{FG} \qquad \text{Applying } \textbf{Theorem 5}\end{aligned}$$

Since we already have $\overset{\frown}{GI}$, we substitute it into the equation to get:

$$\begin{aligned}\overset{\frown}{EG} &= \frac{1}{2}\left(\frac{65\pi}{12}\right) + \frac{25\pi}{12} \text{ m} \\ &= \frac{65\pi}{24} + \frac{50\pi}{24} \text{ m} \\ &= \frac{115\pi}{24} \text{ m} \\ &\approx 15.05 \text{ m}\end{aligned}$$

Step 5: Compare the arc length between E and G and the circumference of Circle 1

To compare them, we take the circumference of Circle 1 and divide it by the arc length of $\overset{\frown}{EG}$.

$$\begin{aligned}\frac{\text{Circle 1 circumference}}{\overset{\frown}{EG}} &= \frac{\frac{65\pi}{3} \text{ m}}{\frac{115\pi}{24} \text{ m}} \\ &= \frac{104}{23} \\ &\approx 4.52\end{aligned}$$

Therefore, the circumference of Circle 1 is approximately 4.52 times larger than the arc length of $\overset{\frown}{EG}$.

Exercise 5

In the following diagram, $\angle AOB = 90°$. Given that $\overset{\frown}{AB} = 2.5$ cm, find the length of $\overline{CO}$, given that $\overline{CO}$ is perpendicular to $\overline{AB}$.